On the universal optimality of the 600-cell: the Levenshtein framework lifted

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Outline

- Minimal Energy, Spherical Harmonics, Gegenbauer Polynomials
- Delsarte-Yudin Linear Programming
- Dual Programming Heuristics
- $1/N$-Quadrature and ULB space
- Levenshtein Framework - ULB Theorem
- Test Functions - Levenshtein Framework Lifted
- The Universality of the 600-cell Revisited

Minimal Energy Problem

- Spherical Code: A finite set $C \subset \mathbb{S}^{n-1}$ with cardinality $|C| = N$.
- $r^2 = |x-y|^2 = 2 - 2\langle x, y\rangle = 2 - 2t$.
- **Interaction potential** $h : [-1, 1) \to \mathbb{R}$
- The h-**energy** of a spherical code $C \subset \mathbb{S}^{n-1}$:

$$E(n, h; C) := \sum_{x,y \in C, y \neq x} h(\langle x, y\rangle),$$

where $t = \langle x, y\rangle$ denotes Euclidean inner product of x and y.

Minimal Energy Problem: Find

$$\mathcal{E}(n, h; N) := \min\{E(n, h; C) \mid C \subset \mathbb{S}^{n-1}, |C| = N\}.$$

Absolutely Monotone Potentials

- **Interaction potential** $h : [-1, 1) \to \mathbb{R}$
- Absolutely monotone potentials:

$$C_+^\infty := \{h \mid h^{(k)}(t) \geq 0, t \in [-1, 1), k \geq 0\}.$$

Examples:

- Newton potential: $h(t) = (2 - 2t)^{-(n-2)/2} = |x - y|^{-(n-2)}$;
- Riesz s-potential: $h(t) = (2 - 2t)^{-s/2} = |x - y|^{-s}$;
- Log potential: $h(t) = -\log(2 - 2t) = -\log|x - y|$;
- Gaussian potential: $h(t) = \exp(2t - 2) = \exp(-|x - y|^2)$;

Spherical Harmonics

- $\mathrm{Harm}(k)$: homogeneous harmonic polynomials in n variables of degree k restricted to $\mathbb{S}^{n-1}$ with

$$r_{k,n} := \dim \mathrm{Harm}(k) = \binom{k+n-3}{n-2}\left(\frac{2k+n-2}{k}\right).$$

- Spherical harmonics (degree k):

$$\{Y_{kj}(x) : j = 1, 2, \ldots, r_{k,n}\}$$

orthonormal basis of $\mathrm{Harm}(k)$ with respect to normalized $(n-1)$-dimensional surface area measure on $\mathbb{S}^{n-1}$.

Spherical Harmonics and Gegenbauer Polynomials

- The Gegenbauer polynomials and spherical harmonics can be defined through the **Addition Formula** ($t = \langle x, y \rangle$):

$$P_k^{(n)}(t) := P_k^{(n)}(\langle x, y \rangle) = \frac{1}{r_k} \sum_{j=1}^{r_k} Y_{kj}(x) Y_{kj}(y), \qquad x, y \in \mathbb{S}^{n-1}.$$

- $\{P_k^{(n)}(t)\}_{k=0}^{\infty}$ orthogonal w/weight $(1-t^2)^{(n-3)/2}$ and $P_k^{(n)}(1) = 1$.

- Gegenbauer polynomials $P_k^{(n)}(t)$ are special types of **Jacobi polynomials** $P_k^{(\alpha,\beta)}(t)$ orthogonal w.r.t. weight $(1-t)^{\alpha}(1+t)^{\beta}$ on $[-1, 1]$, where $\alpha = \beta = (n-3)/2$.

Spherical Designs

- The k**-th moment** of a spherical code $C \subset \mathbb{S}^{n-1}$ is

$$M_k(C) := \sum_{x,y \in C} P_k^{(n)}(\langle x, y \rangle) = \frac{1}{r_k} \sum_{j=1}^{r_k} \sum_{x,y \in C} Y_{kj}(x) Y_{kj}(y)$$
$$= \frac{1}{r_k} \sum_{j=1}^{r_k} \left(\sum_{x \in C} Y_{kj}(x) \right)^2 \geq 0.$$

- $M_k(C) = 0$ if and only if $\sum_{x \in C} Y(x) = 0$ for all $Y \in \mathrm{Harm}(k)$.
- If $M_k(C) = 0$ for $1 \leq k \leq \tau$, then C is called a **spherical** τ**-design** and

$$\int_{\mathbb{S}^{n-1}} p(y)\, d\sigma_n(y) = \frac{1}{N} \sum_{x \in C} p(x), \quad \forall\, p \in \Pi_\tau(\mathbb{R}^n).$$

'Good' potentials for lower bounds

Suppose $f : [-1,1] \to \mathbb{R}$ is of the form

$$f(t) = \sum_{k=0}^{\infty} f_k P_k^{(n)}(t), \qquad f_k \geq 0 \text{ for all } k \geq 1. \tag{1}$$

$f(1) = \sum_{k=0}^{\infty} f_k < \infty \implies$ convergence is absolute and uniform. Then:

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$f(1) = \sum_{k=0}^{\infty} f_k < \infty \implies$ convergence is absolute and uniform. Then:

$$\begin{aligned} E(n, C; f) &= \sum_{x,y \in C} f(\langle x, y\rangle) - f(1)N \\ &= \sum_{k=0}^{\infty} f_k \sum_{x,y \in C} P_k^{(n)}(\langle x, y\rangle) - f(1)N \end{aligned}$$

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$$\begin{aligned} E(n, C; f) &= \sum_{x,y \in C} f(\langle x, y \rangle) - f(1)N \\ &= \sum_{k=0}^{\infty} f_k M_k(C) \qquad - f(1)N \\ &\geq f_0 N^2 - f(1)N = N^2 \left(f_0 - \frac{f(1)}{N} \right). \end{aligned}$$

Delsarte-Yudin LP Bound

Let $A_{n,h} := \{f : f(t) \leq h(t), t \in [-1,1), f_k \geq 0, k = 1, 2, \dots\}$.

Thm (Delsarte-Yudin Lower Energy Bound)

For any $C \subset \mathbb{S}^{n-1}$ with $|C| = N$ and $f \in A_{n,h}$,

$$E(n, h; C) \geq N^2(f_0 - \frac{f(1)}{N}). \tag{3}$$

C satisfies $E(n,h;C) = E(n,f;C) = N^2(f_0 - \frac{f(1)}{N}) \iff$

(a) $f(t) = h(t)$ for $t \in \{\langle x, y\rangle : x \neq y,\ x, y \in C\}$, and

(b) for all $k \geq 1$, either $f_k = 0$ or $M_k(C) = 0$.

Linear program: Maximize D-Y lower bound

Maximizing Delsarte-Yudin lower bound is a linear programming problem.

$$\text{Maximize } N^2\left(f_0 - \frac{f(1)}{N}\right)$$
$$\text{subject to } f \in A_{n,h}.$$

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For a subspace $\Lambda \subset C([-1,1])$, we consider

$$\mathcal{W}(n, N, \Lambda; h) := \sup_{f \in \Lambda \cap A_{n,h}} N^2\left(f_0 - \frac{f(1)}{N}\right). \tag{4}$$

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Usually, $\Lambda = \mathrm{span}\{P_i^{(n)}\}_{i \in I}$ for some finite I,

and we replace $f(t) \leq h(t)$ with $f(t_j) \leq h(t_j)$, $j \in J$ for finite J.

Dual Programming Heuristics

Primal Program

Maximize $c^T x$

subject to $Ax \leq b, x \geq 0$

Dual Program

Minimize $b^T y$

subject to $A^T y \geq b, y \geq 0$

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Minimize $b^T y$

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Primal Maximize $$f_0 - \frac{1}{N} \sum_{i \in I} f_i$$

subject to: $$\sum_{i \in I} f_i P_i^{(n)}(t_j) \leq h(t_j), j \in J, f_i \geq 0.$$

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Maximize $c^T x$

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Minimize $b^T y$

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Primal Maximize $f_0 - \frac{1}{N} \sum_{i \in I} f_i$

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Dual Minimize $\sum_{j \in J} \rho_j h(t_j)$

subject to: $\frac{1}{N} + \sum_{j \in J} \rho_j P_i^{(n)}(t_j) \geq 0, i \in I \setminus \{0\}, \rho_j \geq 0.$

Dual Programming Heuristics - complementary slackness

Add slack variables $\{u_j\}_{j\in J}$ and $\{w_i\}_{i\in I}$.

Primal Maximize $f_0 - \dfrac{1}{N}\displaystyle\sum_{i\in I} f_i$

subject to: $\displaystyle\sum_{i\in I} f_i P_i^{(n)}(t_j) + u_j = h(t_j), j \in J, f_i \geq 0.$

Dual Minimize $\displaystyle\sum_{j\in J} \rho_j h(t_j)$

subject to: $\dfrac{1}{N} + \displaystyle\sum_{j\in J} \rho_j P_i^{(n)}(t_j) - w_i = 0, i \in I \setminus \{0\}, \rho_j \geq 0.$

Complementary slackness condition for Primal Objective=Dual Objective:
$f_i \cdot w_i = 0$, $i \in I$, and $\rho_j \cdot u_j = 0$, $j \in J$.

$1/N$-Quadrature Rules

- For a subspace $\Lambda \subset C[-1,1]$ we say $\{(\alpha_i, \rho_i)\}_{i=1}^k$ with $-1 \leq \alpha_i < 1$, $\rho_i > 0$ for $i = 1, 2, \ldots, k$ is a $1/N$-**quadrature rule exact for** Λ if

$$f_0 = \gamma_n \int_{-1}^{1} f(t)(1-t^2)^{(n-3)/2} dt = \frac{f(1)}{N} + \sum_{i=1}^{k} \rho_i f(\alpha_i), \quad (f \in \Lambda).$$

$$\Longrightarrow f_0 - \frac{f(1)}{N} = \sum_{i=1}^{k} \rho_i f(\alpha_i).$$

$1/N$-Quadrature Rules

- **Example**: Given a spherical τ-design C and $|C| = N$.
  Then $\{\alpha_0 = 1, \alpha_1, \ldots, \alpha_k\} := \{\langle x, y\rangle \mid x, y \in C\}$ and

$$\rho_i := \frac{|\{(x,y) \in C \times C \mid \langle x, y\rangle = \alpha_i\}|}{N^2}, \quad i = 0, \ldots, k$$

  is a $1/N$-QR exact for Π_τ. If $p \in \Pi_\tau([-1,1])$ then for any $y \in \mathbb{S}^{n-1}$ we have

$$\gamma_n \int_{-1}^{1} p(t)(1-t^2)^{(n-3)/2} dt = \int_{\mathbb{S}^{n-1}} p(\langle x, y\rangle) d\sigma_n(x) = \frac{1}{N} \sum_{x \in C} p(\langle x, y\rangle)$$

ULB space

- For $f \in \Lambda \cap A_{n,h}$ and $\{(\alpha_i, \rho_i)\}_{i=1}^k$ exact for Λ:

$$f_0 - \frac{f(1)}{N} = \sum_{i=1}^{k} \rho_i f(\alpha_i) \leq \sum_{i=1}^{k} \rho_i h(\alpha_i),$$

and so

$$\mathcal{W}(n, N, \Lambda; h) \leq N^2 \sum_{i=1}^{k} \rho_i h(\alpha_i), \tag{6}$$

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and so

$$\mathcal{W}(n, N, \Lambda; h) \leq N^2 \sum_{i=1}^{k} \rho_i h(\alpha_i), \tag{7}$$

- with "=" $\iff \left(\begin{array}{c} \exists f \in \Lambda \cap A_{n,h} \text{ such that} \\ f(\alpha_i) = h(\alpha_i), i = 1, \ldots, k \end{array} \right)$
- If equality holds in (7) for **all** $h \in C_+^\infty$, we call Λ (with associated QR) a (n, N)-**ULB space**.

Hermite Interpolation

Suppose $f, h \in C^1([-1,1))$, $f \leq h$ and $f(\alpha) = h(\alpha)$ for some $\alpha \in [-1,1)$.

- If $\alpha > -1$ then $f'(\alpha) = h'(\alpha)$.
- If $\alpha = -1$ then $f'(\alpha) \leq h'(\alpha)$.

If $f(\alpha_i) = h(\alpha_i), i = 1, \dots, k$ and $\alpha_i > -1$, then $2k$ necessary conditions.

Sharp Codes

Observe that a spherical τ-design C yields a $1/|C|$-quadrature rule that is exact for $\Lambda = \Pi_\tau$ with nodes $\{\langle x, y\rangle \mid x \neq y \in C\}$.

Definition

A spherical code $C \subset \mathbb{S}^{n-1}$ is **sharp** if there are m inner products between distinct points in it and C is a spherical $\tau = (2m-1)$-design.

Theorem (Cohn and Kumar, 2007)

If $C \subset \mathbb{S}^{n-1}$ is a sharp code, then C is **universally optimal**; *i.e., C is h-energy optimal for any h that is absolutely monotone on $[-1, 1)$.*

Idea of proof: Show Hermite interpolant to h is in $A_{n,h}$; i.e., Π_τ is a $(n, |C|)$-ULB space.

Levenshtein Framework - $1/N$-Quadrature Rule

- For every fixed $N > D(n, 2k-1)$(the DGS bound) there exists a $1/N$-QR that is exact on $\Lambda = \Pi_{2k-1}$.
- The numbers α_i, $i = 1, 2, \ldots, k$, are the roots of the equation

$$P_k(t)P_{k-1}(s) - P_k(s)P_{k-1}(t) = 0,$$

where $P_i(t) = P_i^{(n-1)/2,(n-3)/2}(t)$ is a Jacobi polynomial and $s = \alpha_k$ is chosen to get weight $1/N$ at node 1.

Universal Lower Bound (ULB)

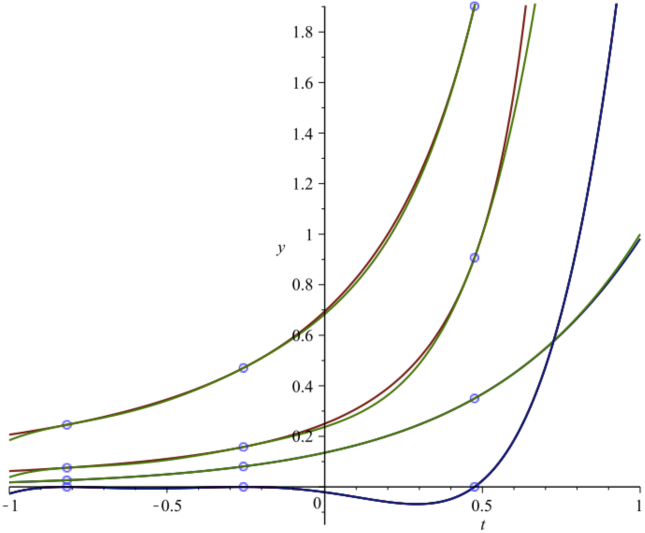
ULB Theorem - (BDHSS, 2016)

Let $h \in C_+^\infty$ and n, k, and N such that $N \geq D(n, 2k-1)$. Then $\Lambda = \Pi_{2k-1}$ is a (n, N)-ULB space with the Levenshtein QR $\{\alpha_i, \rho_i\}_{i=1}^k$; i.e.,

$$\mathcal{E}(n, N, h) \geq N^2 \sum_{i=1}^{k} \rho_i h(\alpha_i).$$

The Hermite interpolants at these nodes are the optimal polynomials which solve the finite LP in the class $\Pi_{2k-1} \cap A_{n,h}$.

Gaussian, Korevaar, and Newtonian potentials



ULB comparison - BBCGKS 2006 Newton Energy

N	Harmonic Energy	ULB Bound	%	N	Harmonic Energy	ULB Bound	%	N	Harmonic Energy	ULB Bound	%
5	4.00	4.00	0.00	25	182.99	182.38	0.34	45	664.48	663.00	0.22
6	6.50	6.42	1.28	26	199.69	199.00	0.35	46	697.26	695.40	0.27
7	9.50	9.42	0.88	27	217.15	216.38	0.36	47	730.75	728.60	0.29
8	13.00	13.00	0.00	28	235.40	234.50	0.38	48	764.59	762.60	0.26
9	17.50	17.33	0.95	29	254.38	253.38	0.39	49	799.70	797.40	0.29
10	22.50	22.33	0.74	30	274.19	273.00	0.43	50	835.12	833.00	0.25
11	28.21	28.00	0.74	31	294.79	293.51	0.43	51	871.98	869.40	0.30
12	34.42	34.33	0.26	32	315.99	314.80	0.38	52	909.19	906.60	0.28
13	41.60	41.33	0.64	33	337.79	336.86	0.28	53	947.15	944.60	0.27
14	49.26	49.00	0.53	34	360.52	359.70	0.23	54	985.88	983.40	0.25
15	57.62	57.48	0.24	35	384.54	383.31	0.32	55	1025.76	1023.00	0.27
16	66.95	66.67	0.42	36	409.07	407.70	0.33	56	1066.62	1063.53	0.29
17	76.98	76.56	0.54	37	434.19	432.86	0.31	57	1108.17	1104.88	0.30
18	87.62	87.17	0.51	38	460.28	458.80	0.32	58	1150.43	1147.05	0.29
19	98.95	98.48	0.48	39	487.25	485.51	0.36	59	1193.38	1190.03	0.28
20	110.80	110.50	0.27	40	514.90	513.00	0.37	60	1236.91	1233.83	0.25
21	123.74	123.37	0.30	41	543.16	541.40	0.32	61	1281.38	1278.45	0.23
22	137.52	137.00	0.38	42	572.16	570.60	0.27	62	1326.59	1323.88	0.20
23	152.04	151.38	0.44	43	601.93	600.60	0.22	63	1373.09	1370.13	0.22
24	167.00	166.50	0.30	44	632.73	631.40	0.21	64	1420.59	1417.20	0.24

Newtonian energy comparison (BBCGKS 2006) - $N = 5 - 64$, $n = 4$.

ULB comparison - BBCGKS 2006 Gaussian Energy

N	Gaussian Energy	ULB Bound	%	N	Gaussian Energy	ULB Bound	%	N	Gaussian Energy	ULB Bound	%
5	0.82085	0.82085	0.0000	25	54.83402	54.81419	0.0362	45	195.4712	195.46	0.0042
6	1.51674	1.469024	3.1460	26	59.8395	59.7986	0.0684	46	204.7676	204.76	0.0049
7	2.351357	2.303011	2.0561	27	65.02733	64.99832	0.0446	47	214.2834	214.27	0.0075
8	3.3213094	3.321309	0.0000	28	70.43742	70.41329	0.0343	48	223.994	223.99	0.0007
9	4.6742772	4.614371	1.2816	29	76.06871	76.0435	0.0332	49	233.9421	233.93	0.0040
10	6.1625802	6.123668	0.6314	30	81.9183	81.88889	0.0359	50	244.0939	244.09	0.0022
11	7.9137359	7.85	0.8517	31	87.99142	87.95307	0.0436	51	254.4665	254.46	0.0028
12	9.8040902	9.780806	0.2375	32	94.26767	94.2326	0.0372	52	265.0585	265.05	0.0049
13	11.975434	11.92615	0.4116	33	100.75	100.7275	0.0223	53	275.8551	275.85	0.0030
14	14.353614	14.28178	0.5005	34	107.4465	107.4377	0.0082	54	286.8694	286.86	0.0020
15	16.90261	16.88487	0.1049	35	114.3862	114.3632	0.0202	55	298.1012	298.1	0.0019
16	19.742184	19.70346	0.1962	36	121.5266	121.504	0.0186	56	309.5522	309.54	0.0030
17	22.795437	22.73703	0.2562	37	128.874	128.86	0.0109	57	321.2188	321.21	0.0041
18	26.046099	25.98526	0.2336	38	136.4529	136.4314	0.0158	58	333.0979	333.08	0.0043
19	29.510614	29.44794	0.2124	39	144.244	144.218	0.0180	59	345.1882	345.18	0.0033
20	33.161221	33.12489	0.1096	40	152.2451	152.2199	0.0165	60	357.497	357.49	0.0033
21	37.051623	37.03121	0.0551	41	160.4628	160.4379	0.0155	61	370.0202	370.01	0.0030
22	137.52	137.00	0.3753	42	168.8894	168.8713	0.0107	62	382.7551	382.75	0.0019
23	41.177514	41.15351	0.0583	43	177.5346	177.5199	0.0083	63	395.7039	395.7	0.0004
24	45.537431	45.49154	0.1008	44	186.3928	186.3839	0.0048	64	408.8804	408.87	0.0021

Gaussian energy comparison (BBCGKS 2006) - $N = 5 - 64$, $n = 4$.

Improvement of ULB and Test Functions

Test functions (Boyvalenkov, Danev, Boumova, '96)

$$Q_j(n, \{\alpha_i, \rho_i\}) := \frac{P_j^{(n)}(1)}{N} + \sum_{i=1}^{k} \rho_i P_j^{(n)}(\alpha_i).$$

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Subspace ULB Improvement Theorem (BDHSS, 2016)

Let $\Lambda \subset C([-1,1])$ be a ULB space with $1/N$-QR $\{(\alpha_i, \rho_i)\}_{i=1}^k$. Suppose $\Lambda' = \Lambda \bigoplus \mathrm{span}\ \{P_j^{(n)} : j \in \mathcal{I}\}$ for some index set $\mathcal{I} \subset \mathbb{N}$. If $Q_j(n, \{\alpha_i, \rho_i\}) \geq 0$ for $j \in \mathcal{I}$, then

$$\mathcal{W}(n, N, \Lambda'; h) = \mathcal{W}(n, N, \Lambda; h) = N^2 \sum_{i=1}^{k} \rho_i h(\alpha_i).$$

If there is $j : Q_j(n, \{\alpha_i, \rho_i\}) < 0$, then $\mathcal{W}(n, N, \Lambda'; h) < \mathcal{W}(n, N, \Lambda; h)$.

Test functions - examples

j	(4, 24)	(10, 40)	(14, 64)	(15, 128)	(7, 182)	(4, 120)
0	1	1	1	1	1	1
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0.021943574	0.013744273	0.000659722	0	0
5	0	0.043584477	0.023867606	0.012122396	0	0
6	0.085714286	0.024962302	0.015879248	0.010927837	0	0
7	0.16	0.015883951	0.012369147	0.005957261	0	0
8	-0.024	0.026086948	0.015845575	0.006751842	0.022598277	0
9	-0.02048	0.02824122	0.016679926	0.008493915	0.011864096	0
10	0.064232727	0.024663991	0.015516168	0.00811866	-0.00835109	0
11	0.036864	0.024338487	0.015376208	0.007630277	0.003071311	0
12	0.059833108	0.024442076	0.01558101	0.007746238	0.009459538	0.053050398
13	0.06340608	0.024976926	0.015644873	0.007809405	0.0065461	0.066587396
14	0.054456422	0.025919671	0.015734138	0.007817465	0.005369545	-0.046646712
15	-0.003869491	0.02498472	0.015637274	0.007865499	0.006137772	-0.018428319
16	0.008598724	0.024214119	0.015521057	0.007815602	0.005268455	0.020868837
17	0.091970863	0.025123445	0.01562458	0.007761374	0.005134928	-0.000422871
18	0.049262707	0.025449746	0.015694798	0.007812225	0.004722806	0.012656294
19	0.035330484	0.024905002	0.015617497	0.00784714	0.003857119	0.006371173
20	0.048230925	0.024837415	0.015589583	0.00781076	0.007863772	0.011244953

Example: ULB's for $N = 24$, $n = 4$ codes

D_4 lattice $= \{v \in \mathbf{Z}^4 \mid \text{sum of components is even}\}$.
C_{24} consists of the 24 minimal length vectors in D_4 lattice (scaled to unit sphere) and is a kissing configuration: $T(C_{24}) = 0.5$.

- C_{24} is 5-design with 4 distinct inner products: $\{-1, -1/2, 0, 1/2\}$.
- Kissing number problem in $\mathbb{R}^4$ – solved by Musin (2003) using modification of linear programming bounds.
- C_{24} is conjectured to be maximal code but not yet proved.
- C_{24} is not universally optimal – Cohn, Conway, Elkies, Kumar (2008); however, D_4 is conjectured to be universally optimal in $\mathbb{R}^4$.

ULB Improvement for $(4, 24)$-codes

For $n = 4$, $N = 24$ Levenshtein nodes and weights (exact for Π_5) are:

$$\{\alpha_1, \alpha_2, \alpha_3\} = \{-.817352..., -.257597..., .474950...\}$$
$$\{\rho_1, \rho_2, \rho_3\} = \{0.138436..., 0.433999..., 0.385897...\},$$

The test functions for $(4, 24)$-codes are:

Q_6	Q_7	Q_8	Q_9	Q_{10}	Q_{11}	Q_{12}
0.0857	0.1600	-0.0239	-0.0204	0.0642	0.0368	0.0598

ULB Improvement for $(4, 24)$-codes

Motivated by this we consider the following space

$$\Lambda := \text{span}\{P_0^{(4)}, \ldots, P_5^{(4)}, P_8^{(4)}, P_9^{(4)}\}.$$

Theorem

The space Λ with $1/24$-QR $\{(\alpha_i, \rho_i)\}_{i=1}^4$ given by

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \approx \{-0.86029, -0.48984, -0.19572, 0.47854\}$$
$$\{\rho_1, \rho_2, \rho_3, \rho_4\} \approx \{0.09960, 0.14653, 0.33372, 0.37847\},$$

is a $(4, 24)$-ULB space. All (relevant) test functions Q_j are now positive so this solves full LP.

Arestov and Babenko (2000) arrive at these nodes, weights in the context of maximal codes.

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LP Optimal Polynomial for $(4, 24)$-code

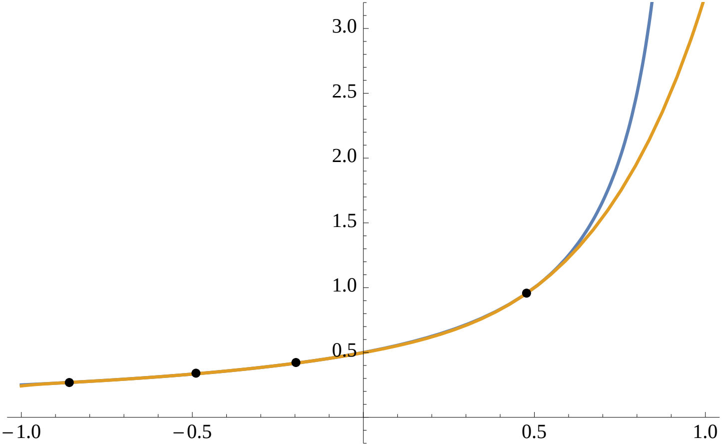

Figure: The $(4, 24)$-code optimal interpolant - Coulomb potential

Sufficient Condition: Partial products

Following ideas from Cohn and Woo (2012) we consider partial products associated with a multi-set $T := \{t_1 \le \cdots \le t_m\} \subset [-1, 1]$

$$p_j(t) := \Pi_{i \le j}(t - t_i).$$

Lemma

Let $\{\alpha_i, \rho_i\}$ be a $1/N$-QR with nodes $-1 \le \alpha_1 < \cdots < \alpha_k$ that is exact for Λ. If $\alpha_1 > -1$, let $T := \{\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_k, \alpha_k\}$, else only take one α_1 once.
Suppose for each $j \le m = |T|$ there exists $q_j \in A_{n,p_j}$ such that $q_j(\alpha_i) = p_j(\alpha_i)$ for $i = 1, \ldots, k$. Then Λ is a (n, N)-ULB space.

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Proof.

For $h \in C_+^\infty$ define

$$f(t) = \sum_{j=1}^{m} h[t_1, \ldots, t_j] q_{j-1}(t),$$

where $h[t_1, \ldots, t_i]$ are the divided differences of h. Then $f \in A_{n,h}$ and $f(\alpha_i) = h(\alpha_i)$, $i = 1, \ldots k$. $\square$

Levenshtein framework lifted - Examples

Dimension	Cardinality	Lev: $\Lambda = \Pi_k$	new: $\Lambda = \Pi_k$
3	14	5	9
3	22	7	11
3	23	7	11
3	32	9	13
3	34	9	13
3	44	11	15
3	47	11	15
3	59	13	17
3	62	13	17
4	24	5	9
4	44	7	11
4	48	7	11
4	120	11	15
5	36	5	9
5	38	5	9

The Universality of the 600-cell Revisited

- C_{600} = 120 points in $\mathbb{R}^4$. Each $x \in C_{600}$ has 12 nearest neighbors forming an icosahedron (Voronoi cells are spherical dodecahedra).
- 8 inner products between distinct points in C_{600}: $\{-1, \pm 1/2, 0, (\pm 1 \pm \sqrt{5})/4\}$.
- 2*7+1 or 2*8 interpolation conditions (would require 14 or 15 design)
- C_{600} is an 11 design, but almost a 19 design (only 12-th moment is nonzero); i.e., $M_k(C_{600}) = 0$ for $k \in \{1, \ldots, 19\} \setminus \{12\}$.

600 cell

- Coxeter (1963), Bőrőczky's (1978) bounds establish maximal code of 600-cell
- Andreev (1999) found polynomial in Π_{17} that shows 600-cell is maximal code.
- Danev, Boyvalenkov (2001) prove uniqueness (of spherical 11-design with 120 points).
- Cohn and Kumar(2007) find family of 17-th degree polynomials that proves universal optimality of C_{600} and they require $f_{11} = f_{12} = f_{13} = 0$; $\Lambda^0_{17} = \Pi_{17} \cap \{P^{(4)}_{11}, P^{(4)}_{12}, P^{(4)}_{13}\}^{\perp}$ with Lagrange condition at -1. Partial product method doesn't work for this family.

600 cell - Levenshtein framework lift, 1^{st} Step

- Levensthein: $n = 4$, $N = 120$, quadrature: 6 nodes exact for Π_{11}:

$$\{\alpha_1, .., \alpha_6\} \approx \{-0.9356, -0.7266, -0.3810, 0.04406, 0.4678, 0.8073\}$$
$$\{\rho_1, .., \rho_6\} \approx \{0.02998, 0.1240, 0.2340, 0.2790, 0.2220, 0.1026\}$$

- Test functions: $Q_{12}, Q_{13} > 0$, $Q_{14}, Q_{15} < 0$.
- Find quadrature rule for $\Lambda_{15} = \Pi_{15} \cap \{P_{12}^{(4)}, P_{13}^{(4)}\}^{\perp}$.

$$\{\beta_1, .., \beta_7\} \approx \{-0.981, -0.796, -0.476, -0.165, 0.097, 0.475, 0.808\}$$
$$C \prod (t - \beta_i) = P_7(t) + C_1 P_6(t) + C_2 P_5(t) + C_3 P_4(t), \quad P_k = P_k^{(\frac{1}{2}, \frac{3}{2})}.$$

- Verify Hermite interpolation works in Λ_{15}.
- New test functions $Q_{12}, Q_{13} > 0$, so this solves LP in Π_{15}.

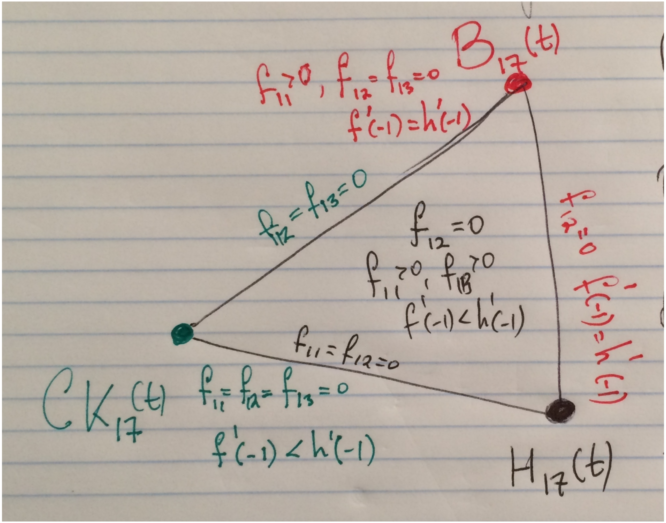
600 cell - Levenshtein framework lift, 2^{nd} Step

- Degree 17. Try $\Lambda_{17}^{1} = \Pi_{17} \cap \{P_{12}^{(4)}, P_{13}^{(4)}\}^{\perp}$, double interpolation at -1. It works.

- Degree 17. Try $\Lambda_{17}^{2} = \Pi_{17} \cap \{P_{11}^{(4)}, P_{12}^{(4)}\}^{\perp}$, double interpolation -1. It works.

- Degree 17. All solutions form triangle.

600 cell - Optimal Triangle in Π_{17}



THANK YOU FOR YOUR ATTENTION !